

Norms:

- **Definition:** Let (X, F) be a linear vector space. A real-valued function is called a norm (and is denoted by $\| \cdot \|$) if the following properties hold:
 - $\|x\| \geq 0$ and $\|x\|=0 \Rightarrow x=0_x$ for each $x \in X$
 - $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ for each $x \in X$ and $\alpha \in F$
 - $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in X$ (triangle inequality)
- **Examples of norms:**
 1. $X = \mathfrak{R}^n$ (X is a vector with n elements)
 - 1-norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$
 - 2-norm or Euclidean norm: $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$
 - ∞ -norm: $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$
 - p -norm: $\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$
 2. $X = C[0, T]$ (X is a continuous function in the given interval $[0, T]$)
 - 1-norm: $\|x\|_1 = \int_0^T |x(t)| dt$
 - 2-norm or Euclidean norm: $\|x\|_2 = \sqrt{\int_0^T |x(t)|^2 dt}$
 - ∞ -norm: $\|x\|_\infty = \max_{t \in [0, T]} |x(t)|$
 - p -norm: $\|x\|_p = \sqrt[p]{\int_0^T |x(t)|^p dt}$
 3. $X \in \mathfrak{R}^{n \times n}$ (Matrix norms)
 - 1-norm: $\|X\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |x_{ij}|$
 - 2-norm or Euclidean norm: $\|X\|_2 = \sqrt{\lambda_{\max}(A^T A)}$
 - ∞ -norm: $\|X\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |x_{ij}|$

Gradient Method – Steepest Descent Method:
$$\begin{bmatrix} x_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \end{bmatrix} - \lambda_k^* \begin{bmatrix} \nabla F(x_k) \end{bmatrix}$$

Newton's Method:
$$\begin{bmatrix} x_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \end{bmatrix} - H(x_k)^{-1} \begin{bmatrix} \nabla F(x_k) \end{bmatrix}$$

Exercises:

1. Compute the 1-norm, ∞ -norm and Euclidean norm (2-norm) of the following matrices:

(a) $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(b) $A_2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

2. Using the first order necessary condition find the minimum points of the following functions:

(a) $f(x) = x_1^2 + x_2^2 - x_1$

(b) $f(x) = x_1^4 + x_1x_2 + \frac{1}{2}x_2^2$

(c) $f(x) = x_1^2 + 2x_2^2 + 4x_1 + 4x_2$

(d) $f(x) = 4x_1^2 - 2x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 + \frac{1}{4}x_2^2$

Verify that the points you find are minimums or not by checking the second order sufficient conditions.

3. Find the point on the line $3x + 2y = 5$ in two-dimensional space closest to the origin when distance is measured by each of the following three norms:

- The 1-norm
- The 2-norm
- The ∞ -norm

4. (a) Let $x = [x_1 \ x_2 \ x_3]^T$ and $F(x) = x_1x_2^2 + x_2x_3^2 + x_3^3$. Compute the Gradient and Hessian of F and find all values x^* for which $\nabla F(x^*) = 0$.

Is the Hessian singular or nonsingular at these values?

- (b) Using the first order necessary conditions ($\nabla F(x^*) = 0$) find a minimum point of the function:

$$F(x_1, x_2, x_3) = 2x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 - 6x_1 - 7x_2 - 8x_3 + 9.$$

Verify that this point is a minimum by checking the second order sufficiency conditions.

5. Write a script to implement the Gradient method – Steepest Descent Method for minimizing the function:

$$F(x_1, x_2) = e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1).$$

Let your initial estimate be something close to the origin. Choose the step-size λ to be a constant. Run a few simulations with different values of λ to see what happens as you vary the step-size from a small to a large value.