

Integer programming:  
an introduction

Alessandro Astolfi

# Outline

- Introduction
- Examples
- Methods for solving ILP
- Optimization on graphs
- LP problems with integer solutions
- Summary

## Introduction

Integer programming is a branch of mathematical programming or optimization.

A general mathematical programming problem can be stated as

$$\max f(x) \quad x \in S \subset \mathbb{R}^n, \quad (1)$$

where  $f$  is called the objective function and it is a function defined on  $S$ , and  $S$  is the so-called constraint set or admissible set.

Every  $x \in S$  is called a feasible solution. Moreover, if there is  $x^*$  such that

$$\infty > f(x^*) \geq f(x)$$

for all  $x \in S$ , then  $x^*$  is called an optimal solution to (1).

The goal of mathematical programming is to establish if an optimal solution exists and to find one, or all, optimal solutions.

## Introduction (cont'd)

An integer programming problem is a mathematical programming problem in which

$$S \subset Z^n \subset \mathbb{R}^n,$$

where  $Z^n$  is the set of all  $n$ -dimensional vectors with integer components.

A mixed integer programming problem is a mathematical programming problem in which at least one, but not all, of the components of  $x \in S$  are required to be integer.

From an applied point of view, it is convenient to regard problem (1) as a model of decision making in which  $S$  represents the set of admissible decisions and  $f$  assigns a utility or profit to each  $x \in S$ .

## Introduction (cont'd)

The problem (1) is called a linear programming (LP) problem if

$$f = cx \quad S = \{x \mid Ax = b, x \geq 0\},$$

where  $c \in \mathbb{R}^{1 \times n}$ ,  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^{m \times 1}$ . Moreover, the inequality  $x \geq 0$  has to be understood componentwise, *i.e.*  $x_i \geq 0$  for all  $i$ .

Note that the set  $S$  is convex, *i.e.* if  $x \in S$  and  $y \in S$  then  $\alpha x + (1 - \alpha)y \in S$  for all  $\alpha \in [0, 1]$ .

A set defined by linear constraints is called a polyhedron or a polytope.

## Introduction (cont'd)

An integer linear programming (ILP) problem is defined as

$$\begin{aligned} \max \quad & cx \\ Ax \quad &= \quad b \\ x \quad &\geq \quad 0 \text{ integer.} \end{aligned} \tag{2}$$

All entries of  $c$ ,  $A$  and  $b$  are assumed integer. (This is equivalent to assuming that they are rational, since multiplication of the objective function by a positive number or of a constraint by any number does not alter the problem.)

Equations (2) provides one possible formulation of an ILP problem.

Alternatively, we may have minimization problems or problems with inequality constraints.

## Introduction (cont'd)

Minimization problems can be rewritten as maximization problems noting that

$$-\min(-f(x)) = \max f(x).$$

Inequality constraints can be converted into equality constraints by adding auxiliary variables. For example

$$ax \leq b \quad \Leftrightarrow \quad ax + s = b \quad s \geq 0,$$

and

$$ax \geq b \quad \Leftrightarrow \quad ax - t = b \quad t \geq 0.$$

The variables  $s$  and  $t$  are known as slack or surplus variables.

## Introduction (cont'd)

The LP problem obtained by dropping the integrality constraint from the ILP problem (2) will be referred to as the corresponding LP problem.

In general, the problem

$$P_1 : \max f(x) \quad x \in S_1$$

is said to be a relaxation of the problem

$$P_2 : \max f(x) \quad x \in S_2$$

if

$$S_1 \supseteq S_2.$$

Similarly,  $P_2$  is said to be a restriction of  $P_1$ .



## Introduction (cont'd)

The concepts of relaxation and restriction are often used in mathematical programming. Note that if  $x^\circ$  is an optimal solution to  $P_1$  and  $x^*$  is an optimal solution to  $P_2$  then

$$f(x^\circ) \geq f(x^*).$$

Moreover, if  $x^\circ \in S_2$  then  $x^\circ$  is an optimal solution to  $P_2$ .

An important special case of the ILP problem is the so-called binary ILP problem described by

$$\begin{aligned} \max \quad & cx \\ Ax \quad &= \quad b \\ x \quad &\geq \quad 0 \quad \text{binary.} \end{aligned} \tag{3}$$

( $x$  binary means  $x_i = 0$  or  $x_i = 1$  for all  $i$ .)

## Examples

Capital budgeting. A firm has  $n$  projects to undertake but, because of budget restrictions, not all can be selected.

Project  $j$  has a present value of  $c_j$ , and requires an investment of  $a_{ij}$  in the time period  $i$ , where  $i = 1, \dots, m$ . The capital available in time period  $i$  is  $b_i$ .

The problem of maximizing the total present value subject to the budget constraints can be written as

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, \dots, m \\ & x_j = 0, 1, \quad j = 1, \dots, n \end{aligned}$$

where  $x_j = 1$  if the project  $j$  is selected and  $x_j = 0$  if the project  $j$  is not selected.

## Examples (cont'd)

Dichotomies. Consider the problem  $\max f(x)$  with  $x \in S$  subject to

$$g(x) \geq 0 \text{ or } h(x) \geq 0. \quad (4)$$

This is in general a difficult problem. However, the dichotomy (4) is equivalent to

$$\begin{aligned} g(x) &\geq \delta \underline{g} \\ h(x) &\geq (1 - \delta) \underline{h} \\ \delta &\text{ binary,} \end{aligned}$$

where  $\underline{g}$  and  $\underline{h}$  are known finite lower bounds on  $g$  and  $h$ . In fact,

$$\delta = 0 \Rightarrow g(x) \geq 0 \text{ and } h(x) \geq \underline{h}$$

$$\delta = 1 \Rightarrow g(x) \geq \underline{g} \text{ and } h(x) \geq 0.$$

## Examples (cont'd)

The fixed charge problem. In general the cost of an activity is a nonlinear function of the activity level  $x$ , given by

$$f(x) = \begin{cases} d + cx & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

If  $d > 0$  and  $f$  is to be minimized, we have the problem

$$\begin{aligned} \min \quad & cx + dy \\ & x \geq 0 \\ & x - uy \leq 0 \\ & y = 0, 1, \end{aligned}$$

where  $y$  is an indicator of whether or not the activity is undertaken, and  $u$  is a known, finite, upper bound for  $x$ . The second constraint guarantees that  $x > 0$  implies  $y = 1$ .

## Examples (cont'd)

The plant location problem. Consider  $n$  customers, the  $j$ -th one requiring  $b_j$  units of a commodity. There are  $m$  locations in which plants may operate to satisfy the demands.

There is a fixed charge of  $d_i$  for opening plant  $i$ , and the unity cost for supplying customer  $j$  from plant  $i$  is  $c_{ij}$ . The capacity of plant  $i$  is  $h_i$ .

The problem is

$$\begin{aligned} \min \sum_{i=1}^m & \left( \sum_{j=1}^n c_{ij} x_{ij} + d_i y_i \right) \\ & \sum_{j=1}^n x_{ij} = b_i \\ & \sum_{j=1}^n x_{ij} - h_i y_i \leq 0 \\ & x_{ij} \geq 0, \quad y_i = 0, 1. \end{aligned}$$

## Examples (cont'd)

The knapsack problem. Suppose  $n$  different types of scientific equipment are considered for inclusion on a space vehicle.

Let  $c_j$  be the scientific value per unit and  $a_j$  the weight per unit of the  $j$ -th type.

If the total weight limitation is  $b$ , the problem of maximizing the total value of the equipment taken is

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ & \sum_{j=1}^n a_j x_j \leq b \\ & x_j \geq 0, \quad \text{integer,} \end{aligned}$$

where  $x_j$  is the number of units of the  $j$ -th type included.

## Methods for solving ILP

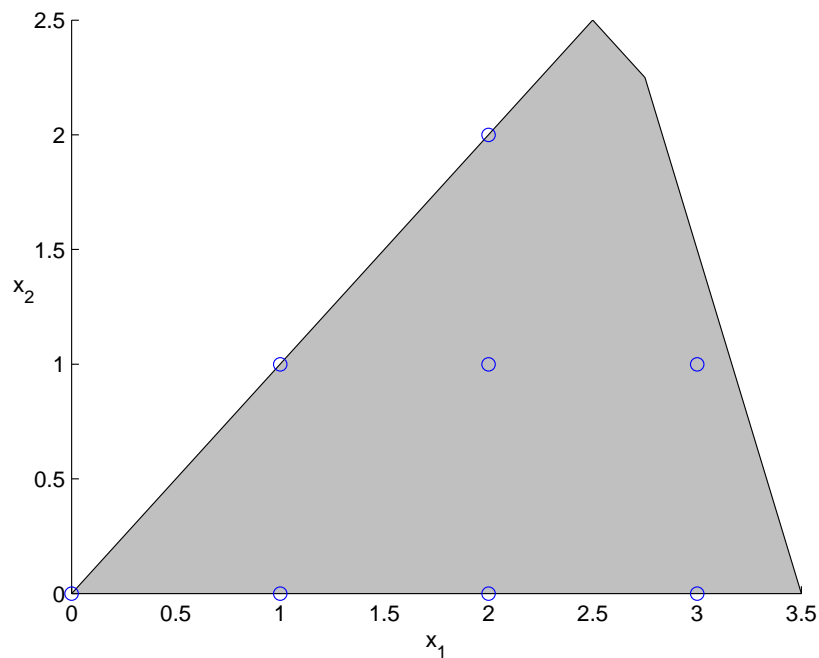
ILP problems can be solved using two basic approaches: enumeration and cutting planes.

To introduce these methods consider the simple problem.

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ & x_1 + x_2 \leq 5 \\ & -x_1 + x_2 \leq 0 \\ & 6x_1 + 2x_2 \leq 21 \\ & x_i \geq 0 \quad \text{integer.} \end{aligned}$$

## Methods for solving ILP (cont'd)

The feasible region is the shaded area in the figure.



The optimum of the relaxed (non-integer) problem is located at  $(\frac{11}{4}, \frac{9}{4})$  with a value of the objective function equal to  $7\frac{3}{4}$ .



## Methods for solving ILP (cont'd)

Enumeration. Without plotting the admissible set it is possible to obtain an upper bound on the number of feasible points.

The first constraint, together with nonnegativity of the  $x_i$ , implies  $0 \leq x_i \leq 5$ .

The third constraint implies  $0 \leq x_1 \leq 3$ .

This limits the feasible points to 24 (16 are infeasible, and 8 are feasible).

By total enumeration one could find the optimal point  $(x_1, x_2) = (3, 1)$ .

## Methods for solving ILP (cont'd)

With some work one can reduce the number of candidate optimal solutions.

Adding the first and second constraints yields  $2x_2 \leq 5$ , which implies  $x_2 \leq 2$ , and reduces the upper bound on the number of feasible points to 12.

Note that the feasible point  $(3, 0)$  yields a value of the objective function equal to 6. Thus every optimal solution should be such that  $2x_1 + x_2 \geq 6$ .

The above, together with  $x_2 \leq 2$  yields  $2x_1 \geq 4$ .

In summary, we have reduced the number of candidate optimal points to 6:

$$(2, 0) \quad (2, 1) \quad (2, 2) \quad (3, 0) \quad (3, 1) \quad (3, 2).$$

## Methods for solving ILP (cont'd)

Of these points,  $(2, 0)$  and  $(2, 1)$  yields a value of the objective smaller than 6.

Moreover, since the non-integer optimum of the objective is  $7\frac{3}{4}$ , it follows that  $2x_1 + x_2 \leq 7$ , which rules out  $(3, 2)$ .

The candidates for optimality have been reduced to

$$(2, 2) \quad (3, 0) \quad (3, 1),$$

from which, by direct computation, one obtains the optimum  $(3, 1)$ .

The main idea of enumeration methods is thus to explore, explicitly or implicitly, a set of integer points containing the set of admissible points.

## Methods for solving ILP (cont'd)

Cutting planes. The idea of cutting planes is to generate a sequence of linear inequalities that *cut out* part of the feasible region of the corresponding LP problem, while leaving the feasible region of the ILP problem unchanged.

If a sufficient number of cutting planes is generated, the ILP problem has the same solution as the corresponding LP problem.

## Methods for solving ILP (cont'd)

Suppose the set  $S = \{x \mid Ax = b, x \geq 0 \text{ integer}\}$  of feasible solutions of an ILP problem is bounded, hence contains a finite number of points.

Define the convex hull of  $S$ , namely

$$S^+ = \{y \mid y = \sum \alpha_i x_i, \alpha \geq 0, \sum \alpha = 1, x_i \in S\}.$$

Then

$$S \subseteq S^+ \subseteq T = \{x \mid Ax = b, x \geq 0\}$$

and the optimal solution of

$$\max cx \quad x \in S$$

can be computed solving

$$\max cx \quad x \in S^+.$$

## Methods for solving ILP (cont'd)

The computation of  $S^+$  is in general very difficult, and involves several cuts.

In practice, a small number of *good* cuts is enough to generate a LP problem with an integer solution, which coincides with the solution of the given ILP problem.

For the considered example, from the optimal solution of the corresponding LP problem one has

$$2x_1 + x_2 \leq 7\frac{3}{4} \Rightarrow 2x_1 + x_2 \leq 7.$$

Moreover

$$2x_1 + x_2 \leq 7 \text{ and } x_2 \geq 0 \Rightarrow 2x_1 \leq 7 \Rightarrow x_1 \leq 3.$$

## Methods for solving ILP (cont'd)

In summary, the problems

$$\max 2x_1 + x_2$$

$$x_1 + x_2 \leq 5 \quad -x_1 + x_2 \leq 0 \quad 6x_1 + 2x_2 \leq 21$$

$$x_i \geq 0 \quad \text{integer}$$

and

$$\max 2x_1 + x_2$$

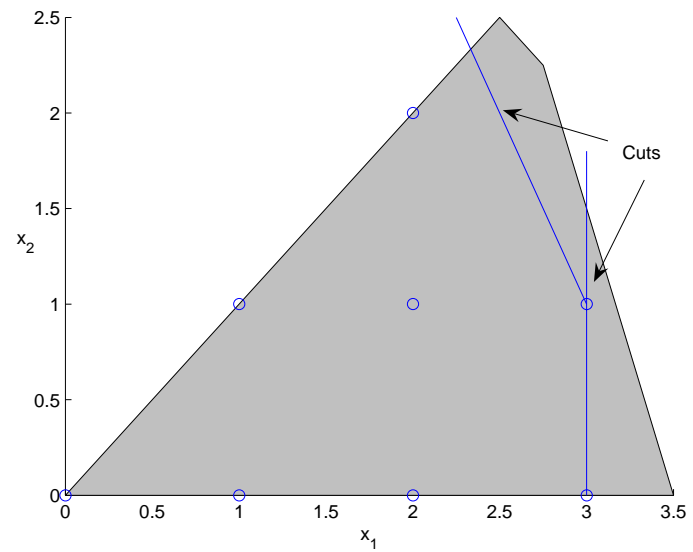
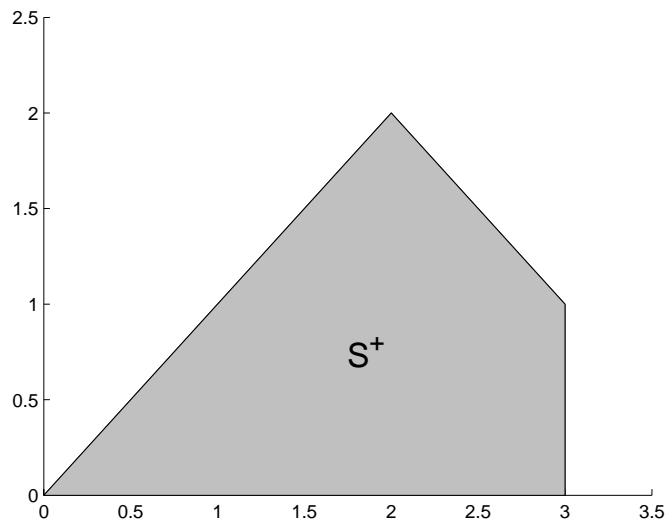
$$x_1 + x_2 \leq 5 \quad -x_1 + x_2 \leq 0 \quad 6x_1 + 2x_2 \leq 21$$

$$x_1 \leq 3 \quad 2x_1 + x_2 \leq 7$$

$$x_i \geq 0$$

have the same optimal solution (the point  $(3, 1)$ ) which is integer.

## Methods for solving ILP (cont'd)





## Optimization on graphs

A significant class of LP and ILP problems is associated with so-called graphs.

Let

$$V = \{1, \dots, m\}$$

be a finite set and let  $Q$  be the set of all ordered pairs of elements of  $V$ , *i.e.*

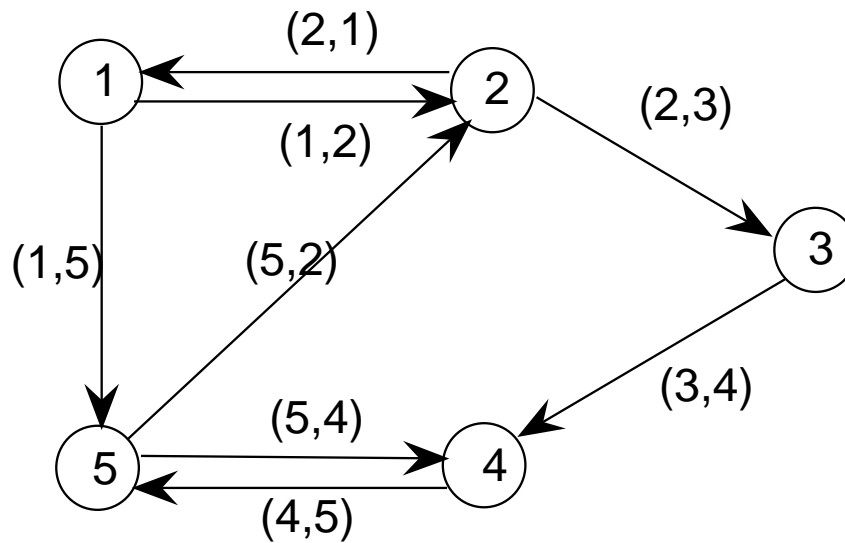
$$Q = \{(i, j) \mid i \in V, j \in V\}.$$

The pair

$$G = (V, E)$$

with  $E \subseteq Q$  is called a directed graph.

The elements of  $V$  are called vertices, those of  $E$  are called directed edges.



$$V = \{1, 2, 3, 4, 5\}$$

$$Q = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), \dots, (4, 5)\}$$

$$E = \{(1, 2), (1, 5), (2, 1), (2, 3), (3, 4), (4, 5), (5, 2), (5, 4)\}$$

## Optimization on graphs (cont'd)

To formulate optimization problems on graphs it is necessary to introduce the following definitions.

Consider a directed graph. Let  $V$  be partitioned as

$$V_1 \text{ (origins)} \quad V_2 \text{ (intermediate points)} \quad V_3 \text{ (destinations)}.$$

For each  $i \in V$  let

$$V(i) = \{j \mid (i, j) \in E\} \quad V'(i) = \{j \mid (j, i) \in E\}.$$

$V(i)$  denotes the set of vertices  $j$  connected to vertex  $i$  by an *outgoing path*.

$V'(i)$  denotes the set of vertices  $j$  connected to vertex  $i$  by an *incoming path*.

## Optimization on graphs (cont'd)

The assignment problem. Consider a directed graph  $E$ .

Assume  $\text{card}V_1 = \text{card}V_3$  and  $V_2 = \emptyset$ .

Assume that each vertex  $i \in V_1$  is connected to all vertices in  $V_3$ .

Consider the problem of minimizing the cost of assigning each vertex of  $V_1$  to a vertex of  $V_3$ .

This problem arises, for example, if one wishes to assign  $m$  men to  $m$  different jobs.

## Optimization on graphs (cont'd)

The assignment problem (cont'd). The problem can be formulated as

$$\begin{aligned} \min \quad & \sum_{i \in V_1} \sum_{j \in V(i)} c_{ij} x_{ij} \\ & \sum_{j \in V(i)} x_{ij} = 1, \quad i \in V_1 \\ & \sum_{i \in V'(j)} x_{ij} = 1, \quad j \in V_3 \\ & x_{ij} \text{ binary,} \end{aligned}$$

where  $c_{ij}$  is the cost of assigning  $i$  to  $j$  and  $x_{ij} = 1$  if  $i$  is assigned to  $j$ .

Interestingly, the corresponding LP problem has an integer optimal solution.

## Optimization on graphs (cont'd)

The shortest path problem. Consider a directed graph  $E$ .

Assume  $V_1 = \{1\}$  and  $V_3 = \{m\}$ .

Let  $c_{ij}$  be the length of edge  $(i, j)$  and define the length of a path as the sum of the lengths of its edges.

Assume all cycles have nonnegative length.

The goal is to find a path from 1 to  $m$  of minimal length.

## Optimization on graphs (cont'd)

The shortest path problem (cont'd). The problem can be formulated as

$$\begin{aligned} \min \quad & \sum_{i \in V} \sum_{j \in V(i)} c_{ij} x_{ij} \\ & \sum_{j \in V(1)} x_{1j} - \sum_{j \in V'(1)} x_{j1} \leq 1 \\ & \sum_{j \in V(i)} x_{ij} - \sum_{j \in V'(i)} x_{ji} = 0, \quad i \in V_2 \\ & \sum_{j \in V(m)} x_{mj} - \sum_{j \in V'(m)} x_{jm} \leq -1 \\ & 0 \leq x_{ij} \leq 1 \text{ integer.} \end{aligned}$$

## LP problems with integer solutions

Consider an LP problem with cost  $cx$  and constraints

$$Ax = b \quad x \geq 0,$$

where  $c$ ,  $A$  and  $b$  have integer entries.

Suppose that the columns of  $A$  are permuted so that

$$A = [B, N]$$

where  $B \in \mathbb{R}^{m \times m}$  is nonsingular, *i.e.*  $\det B \neq 0$ .

The matrix  $B$  is called basis matrix for the LP problem.

There are at most  $\binom{n}{m}$  different basis matrices.



## LP problems with integer solutions (cont'd)

Let  $x = (x_B, x_N)$ , where  $x_B$  is the vector of basic variables associated with the columns of  $B$  and  $x_n$  is the vector of non-basic variables associated with the columns of  $N$ .

Then  $Ax = b$  can be rewritten as

$$Bx_B + Nx_N = b$$

and, since  $B$  is invertible

$$x_B = B^{-1}b - B^{-1}Nx_N.$$

The particular selection

$$(x_B, x_N) = (B^{-1}b, 0)$$

is called a basic solution and if  $x_B \geq 0$  it is called a basic feasible solution.

## LP problems with integer solutions (cont'd)

### **Theorem**

If an LP problem has an optimal solution, it has a basic optimal solution.

In the context of integer programming, one may wonder when an LP problem with integer data has an optimal solution which is integer.

A sufficient condition for a basic solution to be integer is that  $B^{-1}$  is an integer matrix, to this end we introduce the notion of unimodularity.

A square integer matrix  $B$  is called unimodular if  $|\det B| = 1$ .

An integer matrix  $A \in \mathbb{R}^{m \times n}$  is totally unimodular if every square nonsingular sub-matrix of  $A$  is unimodular.

## LP problems with integer solutions (cont'd)

### **Theorem**

Consider the problem  $\max cx$  with constraints  $Ax = b$  and  $x \geq 0$ . If  $A$  is totally unimodular then every basic solution of the problem is integer.

### **Theorem**

Let  $A$  be an integer matrix. Then the following statements are equivalent.

- $A$  is totally unimodular.
- The extreme points (if any) of  $\{x \mid Ax \leq b, x \geq 0\}$  are integer for any integer  $b$ .

## LP problems with integer solutions (cont'd)

### **Theorem**

An integer matrix  $A$  with  $a_{ij} = 0, 1, -1$  for all  $i$  and  $j$  is totally unimodular if

- no more than two nonzero elements appear in each column;
- the rows can be partitioned into two subsets  $Q_1$  and  $Q_2$  such that
  - if a column contains two nonzero elements with the same sign, one element is in each of the subsets;
  - if a column contains two nonzero elements of opposite sign, both elements are in the same subset.

### **Theorem**

The constraint matrices for the assignment and shortest path problems are totally unimodular.

## Summary

We have discussed and formulated integer programming problems.

We have outlined two procedures for the solutions of such problems.

We have considered optimization problems on graphs.

We have discussed the notion of unimodularity and its connection with optimization problems on graphs.