ECE 802 - Optimization of CIS

Tutorial 1 - Linear Programming

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Linear programming describes an important class of optimization problems in which the objective and all the constraint are expressed as linear function, that is.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \cdots, m \end{array}$$

where the vectors c, a_1, $\cdots$, $a_m \in \mathbb{R}^n$ and the scalars b_1, $\cdots$, $b_m \in \mathbb{R}$ are problem parameters that specify the objective and the constraints and x is the so-called decision variable.

Linear programming problems can be solved efficiently, even for problems with thousands of variables. For this reason, non-linear problems are often converted or approximated by linear problems. Note that linear programming problems are convex problems.

Example 1. Solve the linear programming problems formulated below. Note that a two variable problem can be studied and solved geometrically.

$$\begin{array}{ll} \text{maximize} & F = 3x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 4 \\ & 2x_1 + x_2 \leq 5 \\ & -x_1 + 4x_2 \geq 2 \\ & x_1, \; x_2 \geq 0 \end{array}$$

Solution: The optimal solution always lies on the boundary of the feasible (shaded) region in Figure 1. In this case, point A is the optimal solution giving $F = 9$, $x_1 = 1$ and $x_2 = 3$.

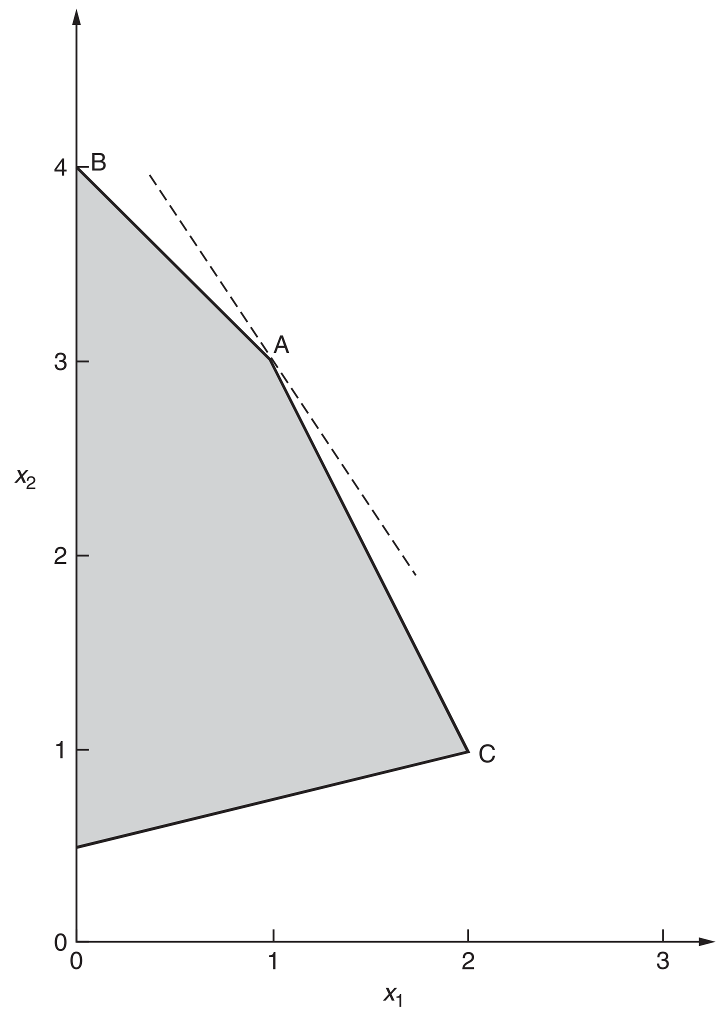

Figure 1: Feasible region of Optimization Problem (4).

Example 2. Solve the linear programming problem of Example 1 using the linprog (Figure 2) optimization toolbox of Matlab (https://www.mathworks.com/help/optim/ug/linprog.html).

linprog

Solve linear programming problems

Syntax

```
x = linprog(f,A,b)
x = linprog(f,A,b,Aeq,beq)
x = linprog(f,A,b,Aeq,beq,lb,ub)
x = linprog(f,A,b,Aeq,beq,lb,ub,options)
x = linprog(problem)
[x,fval] = linprog( __ )
[x,fval,exitflag,output] = linprog( __ )
[x,fval,exitflag,output,lambda] = linprog( __ )
```

Description

Linear programming solver

Finds the minimum of a problem specified by

$$\min_x f^T x \text{ such that } \begin{cases} A \cdot x \le b, \\ Aeq \cdot x = beq, \\ lb \le x \le ub. \end{cases}$$

f, x, b, beq, lb, and ub are vectors, and A and Aeq are matrices.

Figure 2: The linprog optimization toolbox of Matlab.

Solution:

Step 1: Convert the linear programming problem of Example 1 to the standard form:

$$\begin{aligned} \text{minimize} \quad & F_n = -3x_1 - 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 4 \\ & 2x_1 + x_2 \le 5 \\ & x_1 - 4x_2 \le 2 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

Step 2: Code the problem in Matlab and call the solver

```
%% Linear programming problem
% Min Fn=-3x1 - 2x2
% s.t   x1 +  x2 <=  4
%      2x1 +  x2 <=  5
%       x1 - 4x2 <= -2
%       x1       >=  0
%             x2 >=  0
%
% F = Fn;
clear all;

% Objective function
Fn = [-3 -2];

% Bounds of the decision variables
lb = [0 0]; % lower
ub = [inf inf]; % upper

% Inequality constraints
A = [1  1; 2  1; 1 -4];
b = [4; 5; -2];

% Equality constraints (there are no equality constraints)
Aeq = [];
beq = [];

% Call the solver
x = linprog(Fn,A,b,Aeq,beq,lb,ub);

Fn = - 3*x(1) - 2*x(2);  % value of objective
F = - Fn; % value of objective for the maximization problem
```

Results:
$x = [1, 3]$, $F_n = -9$, $F = 9$.

Example 3. Economic Dispatch

Consider a power system with N generating units as shown in Figure 3. In this system there are multiple types of fuel inputs used to operate the power plant (e.g., gas, heavy fuel oil, diesel). Therefore, the generating units have different fuel consumption and characteristics. The economic dispatch determines the power output among the committed generating units to serve the system net load at the minimum total cost.

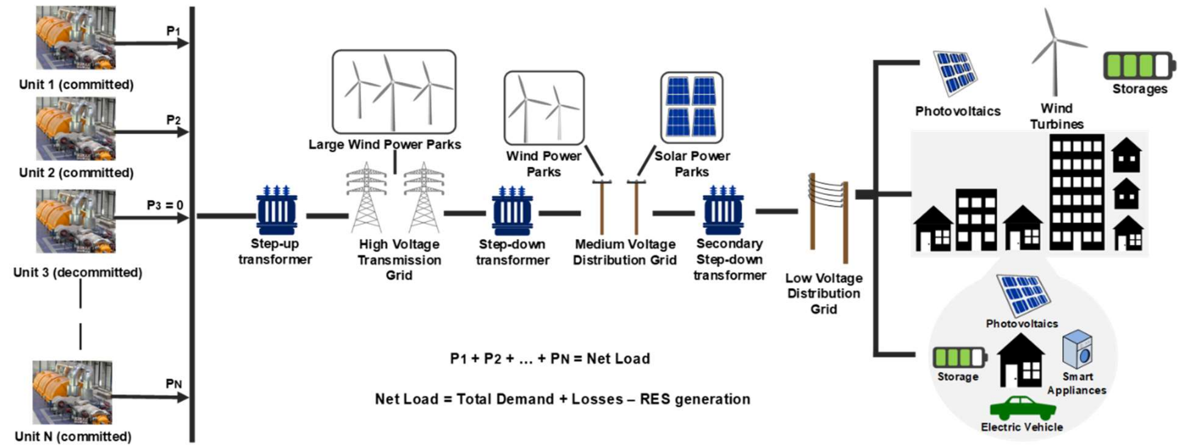

Figure 3: Example of power system including multiple types of power plants

Definition: The economic dispatch is an optimization problem that defines the power output of the committed generating units to minimize the total operational cost of the system, subjects to operational and security constraints.

Linear and Quadratic Cost Functions: The cost function of each generating unit is represented by linear, quadratic, or cubic functions, with the quadratic function the most commonly used description. The cost function determines the operational cost of each unit as a function of the power produced by the unit. The cost function are given by the expression

$$\text{Linear Cost Function:} \quad C_i(P_i) = a_i + b_i P_i, \quad €/\text{h}.$$
$$\text{Quadratic Cost Function:} \quad C_i(P_i) = a_i + b_i P_i + c_i P_i^2, \quad €/\text{h}.$$

where a_i [€/h], b_i [€/MWh] and c_i [€/MWh^2] are the cost coefficients, and P_i [MWh] is the power produced by the generating unit i.

Maximum and Minimum Power: The produced power of a committed unit must be within a minimum ($\check{P}_i$) and a maximum ($\hat{P}_i$) value, based on its technical characteristics, that is

$$\check{P}_i \leq P_i \leq \hat{P}_i, \quad \text{for } i = 1, \cdots, N.$$

Power Balance: The total generation must be equal to the load demand (D), that is

$$\sum_{i=1}^{N} P_i = D.$$

Economic Dispatch vs Unit Commitment: The unit commitment optimization problem defines which units of the power system must be committed (ON) and how much power it has to produce to minimize the system's total operating cost over a time horizon (usually a day ahead). Therefore, the economic dispatch is a sub-problem of the unit commitment. The unit commitment is a mixed-integer programming problem that is very difficult to solve for large-scale systems due to the fact that the computation time increases exponentially as the number of decision variables increases.

Example using economic dispatch via linear programming: Consider a power system with $N = 8$ committed units, that is

- 1 GT unit: $C_i(P_i) = 710 + 60P_i$ €/h, $5 \leq P_i \leq 30$ MW, for i=1,
- 3 ST units: $C_i(P_i) = 670 + 40P_i$ €/h, $25 \leq P_i \leq 70$ MW, for i={2,3,4},
- 2 ICE units: $C_i(P_i) = 150 + 38P_i$ €/h, $7 \leq P_i \leq 28$ MW, for i={5,6},
- 2 ST units: $C_i(P_i) = 870 + 37P_i$ €/h, $50 \leq P_i \leq 140$ MW, for i={7,8}.

where GT = Combustion Gas Turbine , ST = Steam Turbine and ICE= Internal Combustion Engine.

Find the produced power (P_i) of each generating unit $i = \{1, \cdots, N\}$ to minimize the total operational cost of the system, while satisfying the power balance and the generation constraints. Solve the problem for load demand $D = 450$ MW and $D = 580$ MW.

Solution:

Step 1: Problem formulation

Note: The constant coefficients do not affect the solution of the problem. So, they can be removed and added to the total cost after the solution of the optimization problem has been obtained.

$$
\begin{aligned}
\text{minimize} \quad & C_T = \sum_{i=1}^{N} C_i(P_i) \\
\text{subject to} \quad & \sum_{i=1}^{N} P_i = D \\
& \check{P}_i \leq P_i \leq \hat{P}_i, \quad \text{for } i = \{1, \cdots, N\}
\end{aligned}
$$

The objective function, C_T, is the sum of the cost function of the units ($N = 8$) and the objective is to minimize the total cost.

Step 2: Code the problem in Matlab and call the solver

```
%% Economic Dispatch using linear programming problem
% min CTn = 60P1 + 40P2 + 40P3 + 40P4 + 38P5 + 38P6 + 37P4
   +37P8
% s.t P1 + P2 + P3 + P4 + P5 + P6 + P7 + P8 = 450
%       5  <= Pi <= 20,    for i=1
%       25 <= Pi <= 70,    for i={2,3,4}
%       7  <= Pi <= 28,    for i={5,6}
%       50 <= Pi <= 140,   for i={7,8}
%
% CT = CTn + 710 + 3*670 + 2*150 + 2*870
clear all;

% Objective function
F = [60 40 40 40 38 38 37 37];

% Bounds of the decision variables
lb = [5 25 25 25 7 7 50 50]; % lower
ub = [30 70 70 70 28 28 140 140 ]; % upper

% Inequality constraints (there are no inequality const.)
A = [];
b = [];

% Equality constraints
Aeq = ones(1,8); % [1 1 1 ...]
beq = 450;  % load demand

% Call the solver
x = linprog(F,A,b,Aeq,beq,lb,ub);

F_linear_terms = 60*x(1) + 40*x(2) + 40*x(3) + 40*x(4) + 38*x
   (5) + 38*x(6) + 37*x(7) + 37*x(8);
F_constant_terms = 710 + 670*3 + 150*2 + 870*2;

CT = F_linear_terms + F_constant_terms;
```

Results for load demand $D = 450$ MW:
$x = [5,\ 25,\ 25,\ 59,\ 28,\ 28,\ 140,\ 140]$ MW, $C_T = 21908$ €.

Results for load demand $D = 580$ MW:
No feasible solution.

Homework:

An engineering factory can produce five types of product (PR_1, PR_2, $\cdots$, PR_5) by using two production processes: grinding and drilling. After deducting raw material costs, each unit of each product yields the following contributions to profit:

PR_1	PR_2	PR_3	PR_4	PR_5
550 €	600 €	350 €	400 €	200 €

Each unit requires a certain time on each process. These are given below (in hours). A dash indicates when a process is not needed.

	PR_1	PR_2	PR_3	PR_4	PR_5
Grinding	12	20	-	25	15
Drilling	10	8	16	-	-

In addition, the final assembly of each unit of each product uses 20 hours of an employee's time. The factory has three grinding machines and two drilling machines and works a six-day week with two shifts of 8 hours on each day. Eight workers are employed in assembly, each working one shift a day.

The problem is to find how much of each product is to be manufactured so as to maximize the total profit contribution. Formulate the problem as a linear programming problem and use Matlab to solve it.