Tutorial 2

ECE 804- *Eigenvalues and Eigenvectors*

In linear algebra, an **eigenvector** is a nonzero vector that changes by a scalar factor when a linear transformation is applied to it. The corresponding **eigenvalue**, often denoted by s is the factor by which the eigenvector is scaled.

A matrix A acts on vectors $\mathbf{x}$ like a function does, with input $\mathbf{x}$ and output $A\mathbf{x}$. Eigenvectors are vectors for which $A\mathbf{x}$ is parallel to $\mathbf{x}$. In other words:

$$A\mathbf{x} = s\mathbf{x} \tag{1}$$

In this equation, $\mathbf{x}$ is an eigenvector of A and s is an eigenvalue of A. Re-arranging the previous equation,

$$(sI - A)\mathbf{x} = 0 \tag{2}$$

In order for s to be an eigenvalue,

$$\det(sI - A) = 0 \tag{3}$$

The previous equation is called *characteristic equation.* It should be noted that the eigenvalues can be real or complex number. Then, the eigenvectors can be found by applying the eigenvalues on the (1).

Example 1: Determine the eigenvalues and eigenvectors of matrix A.

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then:

$$\begin{aligned} \det(sI - A) &= \begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix} \\ &= s^2 - 1 \\ &= (s-1)(s+1) \end{aligned}$$

The eigenvalues are s_1=1 and s_2=-1. Then we can find the eigenvectors,

From the first eigenvalue s_1=1,

$$\begin{aligned} &A\mathbf{x} = s_1\mathbf{x} \\ &\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\Rightarrow \begin{cases} x_2 = x_1 \\ x_1 = x_2 \end{cases} \xrightarrow{\text{for example}} \mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

From the second eigenvalue s_2=-1

$$
\begin{aligned}
&A\mathbf{x} = s_2\mathbf{x} \\
&\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -1\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&\Rightarrow \begin{cases} x_2 = -x_1 \\ x_1 = -x_2 \end{cases} \xrightarrow{\textit{for example}} \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\end{aligned}
$$

Therefore, $\mathbf{x}_1$ and $\mathbf{x}_2$ are the eigenvectors of matrix A.

Diagonal matrix

A matrix is called diagonal if all the off-diagonal terms are zero. In the general case,

$$
A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}
$$

This matrix has some special properties. So lets find the eigenvalues of this matrix,

$$
\begin{aligned}
&\det(sI - A) = 0 \\
&\Rightarrow \begin{vmatrix} s-a_{11} & 0 & 0 \\ 0 & s-a_{22} & 0 \\ 0 & 0 & s-a_{33} \end{vmatrix} = 0 \\
&\Rightarrow (s-a_{11})(s-a_{22})(s-a_{22}) = 0
\end{aligned}
$$

Therefore, the eigenvalues of a diagonal matrix are actually the diagonal elements. In case a matrix is not diagonal, it can be transformed into a diagonal matrix using some transformations.

If A has n linearly independent eigenvectors, we can put these vectors in columns of a matrix T which is called eigenvalue matrix. Therefore, multiplying matrix A with T we get,

$$
\begin{aligned}
AT &= A\begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_n \end{pmatrix} \\
&= \begin{pmatrix} s_1\mathbf{x}_1 & \cdots & s_n\mathbf{x}_n \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{pmatrix}\begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & \vdots \\ \vdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & s_n \end{pmatrix} = T\tilde{A}
\end{aligned}
$$

where $\mathbf{x}_1$ $\mathbf{x}_2$..$\mathbf{x}_n$ are the eigenvectors of A and s_1..s_n represent the eigenvalues of A. The matrix $\tilde{A}$ is the diagonal matrix of A and can be called eigenvalue matrix.

Considering the previous analysis, we have

$$AT = T\tilde{A}$$

$$\tilde{A} = T^{-1}AT \qquad (4)$$

$$A = T\tilde{A}T^{-1}$$

Important: det(T)≠0 (The inverse of matrix T should exist to be able to estimate the diagonal matrix).

Questions:

1. Find the characteristic equation and the eigenvalues of the following matrices:

a) $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

The characteristic equation is given by $\det(sI - A) = 0$. Hence,

$$\left|\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\right| = 0 \Rightarrow \begin{vmatrix} s-3 & -1 \\ -1 & s-3 \end{vmatrix} = 0$$

$$(s-3)^2 - 1 = 0 \Rightarrow s^2 - 6s + 8 = 0$$

From the characteristic equation, the eigenvalues of A are $s_1=2$ and $s_2=4$.

b) $A = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Characteristic equation:

$$\left|\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}\right| = 0 \Rightarrow \begin{vmatrix} s+1 & 0 & -1 \\ 3 & s-4 & -1 \\ 0 & 0 & s-2 \end{vmatrix} = 0$$

$$(s-2)(s+1)(s-4) = 0$$

Therefore, the eigenvalues are: $s_1=2$, $s_2=-1$ and, $s_3=4$

c) $A = \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}$

Characteristic equation:

$$\left|\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}\right| = 0 \Rightarrow \begin{vmatrix} s-4 & -3 \\ 3 & s-4 \end{vmatrix} = 0$$

$$(s-4)^2+9=0 \Rightarrow s^2-8s+25=0$$

Therefore, the eigenvalues are: s_1=4-3i and s_2=4+3i.

Matrix Exponential

Suppose that we have a first order differential equation with the following form,

$$\frac{dy(t)}{dt}=ay(t) \tag{5}$$

We know the solution,

$$y(t)=y(0)e^{at} \tag{6}$$

where y(0) is the initial condition. Now suppose that we have n differential equations,

$$\frac{d\mathbf{y}(t)}{dt}=A\mathbf{y}(t) \tag{7}$$

where A is a matrix and **y**(t) is a vector. The solution should be,

$$\mathbf{y}(t)=\mathbf{y}(0)e^{At} \tag{8}$$

As can be seen, this solution is a identical with (5). Therefore, e^{At} should be calculated. To do so, let consider the Taylor series for the exponential.

$$e^{x}=1+x+\frac{x^2}{2}+\frac{x^3}{3!}+...+\frac{x^n}{n!} \tag{9}$$

To calculate e^{At} we plug in a matrix instead of a number.

$$e^{At}=I+At+\frac{(At)^2}{2}+....+\frac{(At)^n}{n!} \tag{10}$$

The estimation of a matrix exponential is difficult, and it still be a topic of considerable current research in mathematics and numerical analysis.

$$\begin{aligned} e^{At} &= I+T\tilde{A}T^{-1}t+\frac{(T\tilde{A}T^{-1}t)^2}{2}+....+\frac{(T\tilde{A}T^{-1}t)^n}{n!} \\ &= T(I+\tilde{A}t+\frac{(\tilde{A}t)^2}{2}+...)T^{-1} \\ &= Te^{\tilde{A}t}T^{-1} \end{aligned}$$

Question:

A system is described by the state equations:

$$\dot{x}_1 = 7x_1 + 3x_2$$
$$\dot{x}_2 = 3x_1 + -x_2$$

The state equations can be written in the form $\dot{\mathbf{x}} = A\mathbf{x}$, where A is given below:

$$A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}.$$

Determine the exponential matrix e^{At}.

Answer: