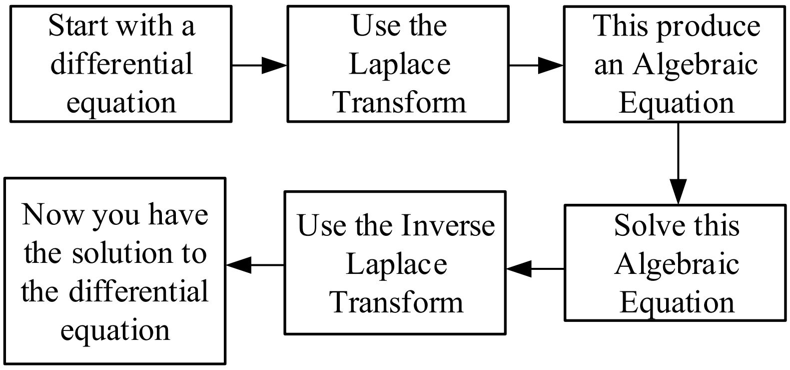
Tutorial 1

ECE 804-Laplace Transform

The Laplace transform is used to solve certain differential equations. The basic idea is:



Definition: The Laplace transform of a general signal *x(t)* is defined as,

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

where s is a complex variable which can be written as $s=\sigma+j\omega$, with σ and ω the real and imaginary parts, respectively.

Example 1: Compute the Laplace transform of $x(t) = e^{-2t}1(t)$

From the definition,

$$X(s) = \int_{-\infty}^{+\infty} e^{-2t}1(t)e^{-st}dt = \int_{0}^{+\infty} e^{-(s+2)t}dt = \left[-\frac{1}{s+2}e^{-(s+2)t}\right]_0^{\infty} = \frac{1}{s+2}$$

Example 2: Compute the Laplace transform of $x(t) = \cos(\omega_o t)1(t)$

$$X(s) = \int_{-\infty}^{+\infty} \cos(\omega_o t)1(t)e^{-st}dt =$$

$$= \int_{0}^{+\infty} \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} e^{-st}dt =$$

$$= \frac{1}{2}\left[\left[-\frac{1}{s-j\omega_o}e^{-(s-j\omega_o)t}\right]_0^{\infty} - \left[\frac{1}{s+j\omega_o}e^{-(s+j\omega_o)t}\right]_0^{\infty}\right] =$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega_o} + \frac{1}{s+j\omega_o}\right] = \frac{s}{s^2+\omega_o^2}$$

Example 3: Compute the Laplace transform of $x(t) = \sin(\omega_o t)1(t)$

$$X(s) = \int_{-\infty}^{+\infty} \sin(\omega_o t)1(t)e^{-st}dt =$$

$$= \int_{0}^{+\infty} \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j} e^{-st}dt =$$

$$= \frac{1}{2j}\left[\left[-\frac{1}{s-j\omega_o}e^{-(s-j\omega_o)t}\right]_0^{\infty} + \left[\frac{1}{s+j\omega_o}e^{-(s+j\omega_o)t}\right]_0^{\infty}\right] =$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega_o} + \frac{1}{s+j\omega_o}\right] = \frac{\omega_o}{s^2+\omega_o^2}$$

Basic Properties of Laplace transform:

- *superposition principle*
  $$z(t) = ax(t) + by(t) \xrightarrow{Laplace} Z(s) = aX(s) + bY(s)$$
- *Time shifting*
  $$x(t) \xrightarrow{Laplace} X(s)$$
  $$x(t-t_0) \xrightarrow{Laplace} e^{-st_0}X(s)$$
- *Laplace transform of 1^{st} –order derivative of x(t)*
  $$y(t) = \frac{dx(t)}{dt} \xrightarrow{Laplace} Y(s) = sX(s) - x(0)$$
- *Laplace transform of 2^{nd} –order derivative of x(t)*
  $$y(t) = \frac{dx^2(t)}{dt^2} \xrightarrow{Laplace} Y(s) = s^2X(s) - sx(0) - \frac{dx(0)}{dt}$$

Example 4: Determine $X_3(s)=X_1(s)-X_2(s)$

$$X_1(s) = \frac{1}{s+1}, X_2(s) = \frac{1}{(s+1)(s+2)}$$

$$\Rightarrow X_3(s) = X_1(s) - X_2(s)$$

$$= \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$

Example 5: Determine the Laplace transform of $e^{-2(t+3)}1(t+3)$

$$e^{-2t}1(t) \xrightarrow{Laplace} \frac{1}{s+2}$$

$$e^{-2(t+3)}1(t+3) \xrightarrow{Laplace} \frac{e^{3s}}{s+2}$$

Poles-zero map

In the most of the practical cases, the Laplace transform is a rational function of *s:*

$$X(s)=\frac{N(s)}{D(s)}$$

- *roots of nominator are called* ***zeros*** (*N(s)=0*)
- *roots of denominator are called* ***poles*** (*D(s)=0*)

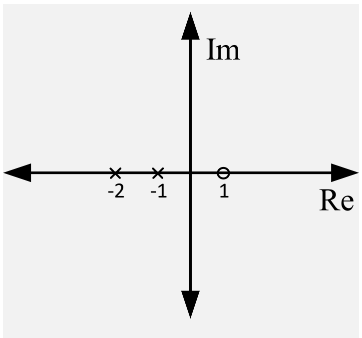
The representation of Laplace transform can be done in the complex plane *s* by placing the poles and zeros.

- Zeros: o
- Poles: x

Example 6: Plot the pole-zero map of the signal $x(t)=3e^{-2t}1(t)-2e^{-t}1(t)$.

Laplace transform of *x(t)* is:

$$X(s)=\frac{3}{s+2}-\frac{2}{s+1}=\frac{s-1}{(s+2)(s+1)}$$



Inverse Laplace transform Definition:

$$x(t)=\frac{1}{2\pi j}\int_{\sigma+j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

The estimation of the inverse Laplace transform is performed using known Laplace transforms and partial fractions and not by the definition.

Example 7:

Determine the inverse Laplace transform of *X(s)*

$$X(s)=\frac{s+3}{(s+1)(s-2)}=\frac{A}{s+1}+\frac{B}{s-2}$$

For estimating A and B,

$$s+3=(s-2)A+(s+1)B$$
$$s=2 \Rightarrow B=\frac{5}{3}$$
$$s=-1 \Rightarrow A=-\frac{2}{3}$$

Therefore, using known Laplace transforms

$$e^{-at}1(t) \xrightarrow{Laplace} \frac{1}{s+a}$$

$$X(s)=\frac{s+3}{(s+1)(s-2)}=\frac{-2/3}{s+1}+\frac{5/3}{s-2}$$

$$x(t)=-\frac{2}{3}e^{-t}1(t)+\frac{5}{3}e^{2t}1(t)$$

Further Questions:

Question 1: Determine the Laplace transform of the following function of time using the table of Laplace transforms (see next page).

$(a)\ x(t)=4\,\mathrm{co9\,ts}(2t)-5t+2e^{3t}1(t)$
$(b)\ x(\mathrm{t})=\mathrm{e}^{-4t}\,1(t)+e^{-5t}(\sin 5t)1(t)$

Question 2: Determine the function of time x(t), for each of the following Laplace transforms.

$(a)\ X(s)=\dfrac{1}{s+1}$
$(b)\ X(s)=\dfrac{1}{s^2+9}$
(c) $X(s)=\dfrac{s+2}{s^2+7s+9}$

Question 3: Consider the signals x(t) and y(t) related through the differential equations.

$$\frac{dx(t)}{dt}=-2y(t)+\delta(t)$$

and

$$\frac{dy(t)}{dt}=2x(t)$$

Determine Y(s) and X(s).

Tutorial 1

ECE804-Laplace Transform-Solutions

Question 1:

(a)

$$\cos(\omega t) \xrightarrow{Laplace} \frac{s}{s^2+\omega^2}$$
$$t \xrightarrow{Laplace} \frac{1}{s^2}$$
$$e^{at}u(t) \xrightarrow{Laplace} \frac{1}{s-a}$$
$$X(s) = 4\frac{s}{s^2+2^2} - 5\frac{1}{s^2} + \frac{2}{s-3} = \frac{6s^4-17s^3+23s^2-20s+60}{s^5-3s^4+4s^3-12s^2}$$

(b)

$$e^{at}u(t) \xrightarrow{Laplace} \frac{1}{s-\alpha}$$
$$e^{at}\sin(\omega t)u(t) \xrightarrow{Laplace} \frac{\omega}{(s-\alpha)^2+\omega^2}$$
$$X(s) = \frac{1}{s+4} + \frac{5}{(s+5)^2+25} = \frac{s^2+15s+70}{s^3+14s^2+90s+100}$$

Question 2:

(a) From table $x(t) = e^{-t}u(t)$

(b) From table $x(t) = \frac{1}{3}\sin(3t)u(t)$

(c) Using partial fraction expansion

$$X(s) = \frac{s+2}{s^2+7s+12} = \frac{A}{s+4} + \frac{B}{s+3}$$
$$A = 2, B = -1$$
$$x(t) = 2e^{-4t}u(t) - e^{-3t}u(t)$$

Question 3:

Taking the Laplace transforms of both sides of the two differential equations, we have

$sX(s) = -2Y(s) + 1$ and $sY(s) = 2X(s)$

Solving for X(s) and Y(s), we obtain

$X(s) = \frac{s}{s^2 + 4}$ and $Y(s) = 2s^2 + 4$

Table of Laplace Transforms

$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}\,dt$
$f + g$	$F + G$
$\alpha f \ (\alpha \in \mathbf{R})$	αF
$\frac{df}{dt}$	$sF(s) - f(0)$
$\frac{d^k f}{dt^k}$	$s^k F(s) - s^{k-1} f(0) - s^{k-2}\frac{df}{dt}(0) - \cdots - \frac{d^{k-1} f}{dt^{k-1}}(0)$
$g(t) = \int_0^t f(\tau)\,d\tau$	$G(s) = \frac{F(s)}{s}$
$f(\alpha t),\ \alpha > 0$	$\frac{1}{\alpha}F(s/\alpha)$
$e^{at} f(t)$	$F(s-a)$
$tf(t)$	$-\frac{dF}{ds}$
$t^k f(t)$	$(-1)^k \frac{d^k F(s)}{ds^k}$
$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
$g(t) = \begin{cases} 0 & 0 \le t < T \\ f(t-T) & t \ge T \end{cases},\ T \ge 0$	$G(s) = e^{-sT} F(s)$

1	$\frac{1}{s}$
δ	1
$\delta^{(k)}$	s^k
t	$\frac{1}{s^2}$
$\frac{t^k}{k!}$, $k \geq 0$	$\frac{1}{s^{k+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2} = \frac{1/2}{s-j\omega} + \frac{1/2}{s+j\omega}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2} = \frac{1/2j}{s-j\omega} - \frac{1/2j}{s+j\omega}$
$\cos(\omega t + \phi)$	$\frac{s \cos \phi - \omega \sin \phi}{s^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$