Norms:

- **Definition**: Let \((X, F)\) be a linear vector space. A real-valued function is called a norm (and is denoted by \(|.||\)) if the following properties hold:
  
  (i) \(|x|| \geq 0 \) and \(|x|=0 \implies x=0\), for each \(x \in X\)
  
  (ii) \(|\alpha \cdot x|| = |\alpha| \cdot |x|\) for each \(x \in X\) and \(\alpha \in F\)
  
  (iii) \(|x + y|| \leq |x|| + |y|\) for each \(x, y \in X\) (triangle inequality)

- **Examples of norms**:
  
  1. \(X = \mathbb{R}^n\) (\(X\) is a vector with \(n\) elements)
     
        1-norm: \(||x||_1 = \sum_{i=1}^{n} |x_i|\)
        
        2-norm or Euclidean norm: \(||x||_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}\)
        
        \(\infty\)-norm: \(||x||_\infty = \max_{1 \leq i \leq n} |x_i|\)
        
        \(p\)-norm: \(||x||_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}\)

  2. \(X = C [0, T]\) (\(X\) is a continuous function in the given interval \([0, T]\))
     
        1-norm: \(||x||_1 = \int_0^T |x(t)| \, dt\)
        
        2-norm or Euclidean norm: \(||x||_2 = \sqrt{\int_0^T |x(t)|^2 \, dt}\)
        
        \(\infty\)-norm: \(||x||_\infty = \max_{t \in [0, T]} |x(t)|\)
        
        \(p\)-norm: \(||x||_p = \sqrt[p]{\int_0^T |x(t)|^p \, dt}\)

  3. \(X \in \mathbb{R}^{n \times n}\) (Matrix norms)
     
        1-norm: \(||X||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |x_{ij}|\)
        
        2-norm or Euclidean norm: \(||X||_2 = \sqrt{\lambda_{\text{max}} (A^T A)}\)
        
        \(\infty\)-norm: \(||X||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |x_{ij}|\)

### Gradient Method – Steepest Descent Method

\[
  x_{k+1} = x_k - \lambda_k^* \left[ \nabla F(x_k) \right]
\]

### Newton’s Method

\[
  x_{k+1} = x_k - H(x_k)^{-1} \left[ \nabla F(x_k) \right]
\]
Exercises:

1. Find the point on the line $3x + 2y = 5$ in two-dimensional space closest to the origin when distance is measured by each of the following three norms:
   a. The 1-norm
   b. The 2-norm
   c. The $\infty$-norm

2. (a) Let $x = [x_1 \ x_2 \ x_3]^T$ and $F(x) = x_1x_2^2 + x_2x_3^2 + x_3^3$. Compute the Gradient and Hessian of $F$ and find all values $x^*$ for which $\nabla F(x^*) = 0$.
   Is the Hessian singular or nonsingular at these values?

   (b) Using the first order necessary conditions ($\nabla F(x^*) = 0$) find a minimum point of the function:
   $F(x_1, x_2, x_3) = 2x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 - 6x_1 - 7x_2 - 8x_3 + 9$.
   Verify that this point is a minimum by checking the second order sufficiency conditions.

3. Write a MATLAB program to implement the Gradient method – Steepest Descent Method for minimizing the function:
   $F(x_1, x_2) = e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1)$.
   Let your initial estimate be something close to the origin. Choose the step-size $\lambda$ to be a constant. Run a few simulations with different values of $\lambda$ to see what happens as you vary the step-size from a small to a large value.